PROBLEMS—NOTES.

7. The following problem is discussed in *Revista Matemática Hispano-Americana*, 1920, pages 190–191, 226–228: "What is the maximum number of spheres a decimeter in diameter that can be placed in a box in the form of a cube a meter along any interior edge?" It is found that this number is 1254. Professor R. P. Baker's problem, 501 (geometry), twice proposed in this MONTHLY [*1916*, 341; *1919*, 414], but not yet solved, may be recalled in this connection: "Find the minimum amount of lumber one inch thick required to pack a gross of spheres three inches in diameter in a rectangular box." ARC.

8. Perfect Numbers. A number which equals the sum of its divisors other than itself is called perfect. Euclid [about 300 B.C.] proved that if

$$p = 1 + 2 + 2^2 + \cdots + 2^{n-1}$$

is a prime, $2^{n-1} \cdot p[= 2^{n-1}(2^n - 1)]$ is a perfect number. So far as known, the first six perfect numbers are: 6, 28, 496, 8128, 33550336, and 8589869056. A posthumous paper of L. Euler [1707–1783] contains the proof that every even perfect number is of Euclid's type.[1] He proved also that every odd perfect number must be of the form $r^{4\lambda+1}P^2$, where r is a prime of the form $4n + 1$. No odd perfect number is known. It has been proved that there is no odd perfect number less than two million, or with less than five distinct prime factors.[2]

Mersenne stated, in effect, in the preface of his *Cogitata Physico-Mathematica*, Paris, 1644, that the first eleven perfect numbers are $2^{n-1}(2^n - 1)$ for $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 167, 257$. Since p can be a prime only when n is a prime the problem of verifying the statement, so far as even perfect numbers are concerned, is equivalent to that of determining when p is prime for the 56 prime values of n less than 258. This has been effected for 45 of the prime values and the untested cases are for $n = 137, 139, 149, 157, 167, 193, 199, 227, 229, 241$, and 257. Mersenne's statement has been shown to be incorrect in at least five cases, namely when $n = 61, 89, 107, 127$, and 67. In the first four of these cases p has been shown to be prime. F. N. Cole found the two prime factors of p when $n = 67$, *Bulletin of the American Mathematical Society*, volume 10, page 137, 1903. America's further contribution to the discussion of perfect numbers was made through R. E. Powers who verified, in 1911, that p is prime when $n = 89$; proved, in 1914, that for $n = 107$, p is prime; and, in 1916, showed that for $n = 103$ or 109, p is composite.[3] According to *Sphinx-Oedipe*, July, 1914, and February, 1920, E

[1] The neatest proof of this was given in six lines by L. E. Dickson, in this MONTHLY, *1911*, 109.

[2] In a paper on perfect numbers by Benjamin Peirce, "Mathematical Instructor in Harvard University," *The New York Mathematical Diary*, no. 13, vol. 2, pp. 267–277, 1832, it is shown that there can be no odd perfect number "included in the forms a^r, a^rb^s, $a^rb^sc^t$, where a, b, and c are prime numbers and greater than unity." L. E. Dickson, in the work referred to on the opposite page, does not mention this paper. He does record: in 1844 "V. A. Lebesgue stated that he had a proof that there is no odd perfect number with fewer than four distinct prime factors." We now see that an American mathematician gave a *proof* of this theorem twelve years earlier.

[3] In his inaugural lecture before the University of Oxford (*Some Famous Problems of the Theory of Numbers and in particular Waring's Problem*, Oxford, 1920) G. H. Hardy remarked: "We have seen this ["tendency to exaggerate the profundity implied by the enunciation of a

Fauquembergue showed p prime for $n = 107$ and 127. Hence to sum up, p is known to be prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107$, and 127; and therefore 12 perfect numbers are known as such. A. Cunningham announced (*Report, British Association for the Advancement of Science*, 1895, p. 614) 12 other values of n ranging in value from $n = 317$ to $n = 132{,}019$, for each of which p is composite. Niewiadomski found a similar result for $n = 761$.

"It must always," wrote Sylvester in 1888, "stand to the credit of the Greek geometers that they succeeded in discovering a class of perfect numbers which in all probability are the only numbers which are perfect." (*Collected Mathematical Papers*, volume 4, 1912, page 589.) Sylvester referred also to the question of the non-existence of odd perfect numbers as "a *problem of the ages* comparable in difficulty to that which previously to the labors of Hermite and Lindemann . . . environed the subject of the quadrature of the circle" (*ibid.*, page 626).

The authoritative reference work which should be consulted by those desiring to learn what has been done in connection with perfect numbers is L. E. Dickson, *History of the Theory of Numbers*, volume 1, 1919. W. R. Ball's statements in the chapter on "Mersenne's numbers," and elsewhere in his *Mathematical Recreations and Essays* (fifth edition, 1911), should be carefully checked with Dickson's work. ARC.

9. In *Norsk Matematisk Tidsskrift*, the official organ of the Norwegian Mathematical Society, the following question was proposed in 1919, volume 1, page 80: "Usually x^y is greater or less than y^x; but for special values of x and y they are equal to one another, e.g., $2^4 = 4^2$. What are other values of x and y for which x^y and y^x are equal?" A reply by T. Nagel appears in the third number of the *Tidsskrift* for 1920, pages 93–94.

Mr. Philip Franklin, of Princeton University, has kindly informed me that in seeking the equations of all curves having the property that their evolutes are equal curves [compare *1920*, 303–306] he was led to a solution involving the roots of the transcendental equation $x = e^{ax}$. If x and y are two roots of this equation[1]

$$x^y = y^x = e^{axy}.$$

theorem"] even in the case of Fermat, a mathematician of a class to which Waring had not the slightest pretensions to belong, whose notorious assertion concerning 'Mersenne's numbers' has been exploded, after the lapse of over 250 years, by the calculations of the American computer Mr. Powers." Such a statement from such an authority, and on such an occasion, was extraordinary. So far as known Fermat made no erroneous statement whatever in connection with Mersenne's numbers. It is learned that the fanciful conjectures of Ball were Professor Hardy's only authority. *Mersenne's* inaccuracies and Mr. Powers's correction of *them* have been noted above.

[1] K. Schwering showed in 1878 (*Zeitschrift für Mathematik und Physik*, vol. 23, pp. 339–343) that solutions of the equation $x^y = y^x$ were obtained by taking any two solutions of the equation $a^x = x$. He showed also that for this latter equation there are an infinite number of solutions given by complex values for x. In 1896–1897, E. M. Lémeray found a number of results in connection with this equation (*Nouvelles Annales de Mathématiques*, vol. 16, pp. 548–556; and vol. 17, pp. 54–61). Among these were the following: when a is between 0 and 1, the equation $a^x = x$ has a real root between 0 and 1; when a is between 1 and $e^{1/e}$ there is a real root betwee 1 and e and a second real root between e and ∞ [$e^{1/e} = 1.444667\cdots$; for $a = \sqrt[3]{3} = 1.442249\cdots$ we have as roots of $(\sqrt[3]{3})^x = x$, $x_1 = 3$ and $x_2 = 2.478055\cdots$ (E. Heis, *Sammlung von Beispielen*

He showed in this MONTHLY, *1916*, 233–237, that the polar equation of $x^y = y^x$ is $r = \sec\theta\cdot(\tan\theta)^{1/(\tan\theta - 1)}$ and that its parametric equations are $x = m^{1/(m-1)}$ and $y = m^{m/(m-1)}$.

But the discussion of the curve goes back much farther; indeed these parametric equations were given by Euler in his *Introductio in analysin infinitorum*, tome 2, Lausanne, 1748, p. 294 (French edition, 1797, p. 297). Setting $m - 1 = 1/u$ he gave also the form $x = \left(1 + \frac{1}{u}\right)^u$ and $y = \left(1 + \frac{1}{u}\right)^{u+1}$ and continued: "Thus the curve has, in addition to the line EAF, the branch RS which converges towards the lines AG and AH as asymptotes[1] and of which AF will be a diameter. Further the curve will cut the line AF at the point C where $AB = BC = e$, e denoting the number whose logarithm is unity. The equation furnishes also an infinity of separate points which with the line EF and the curve RCS exhaust those defined by it. There is, then, an infinity of numbers x and y which taken two and two satisfy the equation $x^y = y^x$; such are the following numbers, among these which are rational: $x = 2$, $y = 4$; $x = 9/4$, $y = 27/8$; $x = 64/27$, $y = 256/81$; $x = 625/256$, $y = 3125/1024$; etc." T. Nagel proved that: $x = 2$, $y = 4$, is the only pair of positive *integers* ($y > x$) satisfying the equation. C. Herbst showed[2] in 1909 that, if $|y| > x$, $x = 2$, $y = 4$ and $x = -2$, $y = -4$ are the only integers satisfying the relation. These negative values were, apparently, overlooked by Daniel Bernoulli when writing to Goldbach[3]

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und Aufgeben aus $\cdots$ *Arithmetik und Algebra*. 75. Aufl., Köln, 1888, p. 371).] For $a = e^{1/e}$ there is a double root whose value is e. When $a = e$, Cauchy found (*Leçons sur le calcul différentiel*, Paris, 1829, leçon 14) $x = 0.3181317 \pm 1.3372357\sqrt{-1}$. By another method Stern got (Crelle's *Journal*, vol. 22, 1841, p. 59) $x = 0.318133 \pm 1.337238\sqrt{-1}$. The expression converging to the value of a root for a given a is $a^{a^{a^{\cdot^{\cdot}}}}$. This expression was studied by G. Eisenstein and F. Woepcke (Crelle's *Journal*, vol. 28, 1844, pp. 49–53; vol. 42, 1851, pp. 83–90). Certain errors of Eisenstein were corrected by P. L. Seidel who found, in effect, the Lémeray results noted above (*Abhandlungen der kgl. Baierischen Akademie der Wissenschaften*, zweite classe, vol. 11, 1870). The rôle that the equation $\omega^\xi = \xi$ plays in Cantor's theory of transfinite numbers will be recalled; see, for example, G. Cantor, *Mathematische Annalen*, vol. 49, 1897, pp. 242–246 (also English translation by P. E. B. Jourdain, Open Court Publ. Co., 1915, pp. 195–201).

Reference may be given also to: Hessel, (1) "Ueber die Bedingung unter welcher $a^x > x$ ist," (2) "Ueber das merkwürdige Beispiel einer zum Theil punctirt gebildeten Curve das der Gleichung entspricht: $y = \sqrt[x]{x}$," *Archiv der Mathematik und Physik*, vol. 14, 1850, pp. 93–96, 169–187; H. Scheffler, "Ueber die durch die Gleichung $y = \sqrt[x]{x}$ dargestellten Curven," *Archiv d. Math. u. Physik*, vol. 16, 1851, pp. 133–137; L. Oettinger. "Ueber den grossten Werth von $\sqrt[x]{x}$ und einige damit zusammenhängende Sätze," *Archiv d. Math. u. Physik*, vol. 42, 1864, pp. 106–113; and to L. Moreau: (1) *Analyse ou nombre de solutions et fixations des racines rémarquables de l'équation* $a^x = x$. (Brussels, 1900, 8vo, 16 pages); (2) "Variation du rapport a^z/z d'un nombre à son logarithme," *Journal de Mathématiques Spéciales*, vol. 25, pp. 170–173, 1900. Nomographic discussions of $a^x = bx$ and $a^x = x^b$ are given in S. Brodetsky, *A First Course in Nomography*, London, 1920, pp. 130–133.

[1] Euler's error here in stating that the coordinate axes are asymptotes (instead of $x = 1$ and $y = 1$) does not appear to have been previously remarked. The four asymptotes ($x = \pm 1$, $y = \pm 1$) were first correctly given in 1913 by A. M. Nesbitt, *Mathematical Questions and Solutions from 'The Educational Times,'* n.s. vol. 23, pp. 77–78, where the curve is plotted and discussed. Somewhat fuller considerations of a similar character were given by E. J. Moulton in this MONTHLY, *1916*, 233–237.

[2] *Unterrichtsblätter für Mathematik und Naturwissenschaften*, Jahrgang 15, pp. 62–63.

[3] *Correspondance Mathématique et Physique* (Fuss), vol. 2, 1843, p. 262.

June 29, 1728: "Je finirai par un problème qui m'a paru fort curieux et que j'ai résolu. Le voici: Trouver deux nombres inégaux x et y tels que $x^y = y^x$. Il n'y a qu'un cas ou ces nombres soient entiers, savoir $x = 2$ et $y = 4$ (car $2^4 = 4^2$), mais on peut donner une infinité de nombres rompus qui satisfont au problème. Il y a aussi d'autres espèces de quantités dont je ne dirai rien." [1] In reply to this on January 31, 1729, Goldbach wrote as follows[2] (*l.c.* pp. 280–281): "Je ne trouve pas la moindre difficulté à faire voir que, dans l'équation $x^y = y^x$, les nombres x et y ne peuvent être entiers à moins que l'un ne soit $= 2$, et l'autre $= 4$, et que, pour les nombres rompus, on peut donner une infinité de solutions. Voici comment je m'y prends: Je fais $y = ax$, donc $x^{ax} = a^x x^x$ et enfin $x = a^{1/(a-1)}$. Or, il est visible que x ne peut être un nombre entier que dans la supposition de $a = 2$; car si a est un nombre entier plus grand que 2, on voit d'abord que x devient irrationel; d'un autre côté, a étant un nombre rompu, toutes ses puissances seront autant de nombres rompus, et par conséquent x ne peut être un nombre entier; mais pour exprimer la valeur de x par des nombres rompus, il n'y a qu'à faire

$$x = f^{g/(f-g)} : g^{g/(f-g)}$$

où f et g soient des nombres entiers." [3] ARC.

PROBLEMS—SOLUTIONS

2791 [1919, 414]. A cup of wine is suspended over a cup of equal capacity full of water; through a small hole in the bottom, the wine drips into the water, and the mixture drips out at

[1] Concerning this passage Cantor remarks (*Vorlesungen über Geschichte der Mathematik*, vol. 3_2, 1901, p. 610): "Man wird nach diesem Schlussworte wohl oder übel annehmen müssen, dass Bernoulli an complexe Auflösungen dachte." One must agree with Eneström (*Bibliotheca Mathematica*, vol. 13 (3), p. 270) that this surmise is "höchst unwahrscheinlich." The natural interpretation is that Bernoulli referred to the infinite number of irrational values of x and y which are obtained from Euler's equations given above, when u is not integral.

[2] The argument of Herbst is very similar.

[3] Other discussions of the relation $x^y = y^x$ are: T. Wittstein, *Archiv der Mathematik und Physik*, vol. 6, pp. 154–162, 1845; I. L. A. Lecointe, *Nouvelles Annales de Mathématiques*, vol. 11, pp. 187–189, 1852; M. Cantor, *Zeitschrift für mathematische und naturwissenschaftlichen Unterricht*, vol. 9, pp. 163–164, 1878; L. F. Marrecas Ferreira, *Jornal de Sciencias Mathematicas e Astronomicas*, vol. 2, pp. 165–166, 1880; M. Luxemberg, *Archiv der Mathematik und Physik*, vol. 66, pp. 332–334, 1881; D. Besso, U. Danielli, and L. Carline, *Periodico di Matematica per l'Insegnamento Secondario*, vol. 5, pp. 12–15, 115–117, 117–119, 1890; A. Flechsenhaar, and R. Schimmack, *Unterrichtsblätter für Mathematik und Naturwissenschaften*, vol. 17, pp. 70–73, 1911 and vol. 18, pp. 34–35, 1912; A. Tanturri, *Periodico di Matematica* . . ., vol. 30, pp. 186–187, 1915.

In Nouvelles Annales de Mathématiques, 1876, Moret-Blanc proved (pp. 44–46) that the only positive integral solutions of the equation $x^y = y^x + 1$, are $y = 0$, x arbitrary; $y = 1$, $x = 2$; $y = 2$, $x = 3$; also Meyl proved (pp. 545–547) that the only positive integral solutions of the equation $(x + 1)^y = x^{y+1} + 1$, are $x = 0$, y arbitrary; $x = 1$, $y = 1$; $x = 2$, $y = 2$. Landau showed (*L'Intermédiaire des Mathématiciens*, 1901, pp. 151–152) that the solutions found by Moret-Blanc for the equation $x^y = y^x + 1$ are the only positive rational solutions. This equation may be obtained from the simultaneous equations $x^y = 3$, $y^x = 2$ for which E. Heis gave the approximate solution $x = 2.23925113$, $y = 1.36280365$ (*Sammlung von Beispielen und Aufgaben aus . . . Arithmetik und Algebra.* 75. Aufl., Köln, 1888, p. 372; *Ausführliche Auflösuug der in Dr. Ed. Heis' Sammlung von Biespielen enthaltenen Aufgaben*, Dritter Theil, Bonn, 1880, pp. 386–388).

In *Messenger of Mathematics*, A. Cunningham discussed the factorization of $x^y - y^x$, x and $y > 1$, and x prime to y (April, 1916, vol. 45, pp. 185–192), and of $2^x - x$, x positive (May, 1917, vol 47, pp. 1–38).